Differential Geometry I: Curves and surfaces

A. Standard exercises:

7.1 The goal of this exercise is to prove that the differential of the determinant map

$$\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$$

at $A \in M_n(\mathbb{R})$ is the linear map

$$d(\det)_A : M_n(\mathbb{R}) \to \mathbb{R}$$

given by

$$d(\det)_A(H) = \mathrm{Tr}(\mathrm{Cof}(A)^\top H),$$

where $\mathrm{Cof}(A)$ is the cofactor matrix of A. Recall also from linear algebra that, when A is invertible, we have $A^{-1} = \frac{1}{\det(A)}\mathrm{Cof}(A)^\top$.

We will proceed in three steps:

1. First, prove the formula above or the case $A = I$.
2. Next, assume that $A \in GL_n(\mathbb{R})$, i.e., that A is invertible.
3. Finally, establish the formula for a general matrix $A \in M_n(\mathbb{R})$, using the fact that the matrix $A + tI$ is invertible for t small enough.

   (This is a generally useful trick for identities involving matrices: When required to establish a matrix identity which involves only continuous expressions with respect to the matrix coefficients, it suffices to establish the same identity only for a dense subset of matrices; the subsets of invertible matrices and of diagonalizable matrices are both dense in $\mathcal{M}_n(\mathbb{R})$.)

7.2 Let $\gamma : I \to \mathbb{R}^2$ be a regular plane curve of class C^3 and $r \geqslant 0$. The *parallel curve* to γ at distance r is defined by

$$\gamma_r(t) = \gamma(t) + rN_\gamma(t)$$

(where $N_\gamma = J(T_\gamma)$ is the oriented normal vector field to γ).

(a) Compute the oriented curvature $\kappa_r(t)$ of the parallel curve γ_r (in terms of r and t).

(b) Show that the function $r \mapsto \kappa_r$ satisfies the Riccati differential equation

$$\frac{\partial \kappa}{\partial r} = \kappa^2.$$

(c) Suppose that

$$q = \inf_{t \in I} \frac{1}{|\kappa(t)|} > 0.$$

Show that the map

$$f : (-\varepsilon, \varepsilon) \times I \to \mathbb{R}^2, \quad f(r,t) = \gamma_r(t)$$

is an immersion for all $\varepsilon \leqslant q$.

(d) Make this explicit for the circle of radius a centered at 0.

(e) Explain why the statement in point (c) fails for $\varepsilon > q$.

Remark. This exercise shows in particular that locally, in a neighborhood of the curve, one can construct a curvilinear coordinate system in which one coordinate is the arc length of the curve and the other is the signed distance to the curve. These coordinates are called *Fermi coordinates*.

7.3 Let $I \subset \mathbb{R}$ be a non-empty open subset. Show that, for any $n \geqslant 2$, there exists no homeomorphism $f : I \to \mathcal{U}$ to an open subset $\mathcal{U}$ of $\mathbb{R}^n$. *(Hint: Check how connecting pairs of points with continuous paths is affected after removing a point from I and from $\mathcal{U}$.)*

Remark. In general, using analogous arguments one can show that a non-empty open set in $\mathbb{R}^n$ is homeomorphic to an open set of $\mathbb{R}^m$ only when $n = m$.

7.4 **(a)** Is the cone

$$\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = (az)^2\}$$

(with $a \neq 0$) a submanifold of $\mathbb{R}^3$? What about $\mathcal{C} \setminus 0$?

(b) Prove that there exists a differentiable submanifold of $\mathbb{R}^6$ that is homeomorphic to $S^2 \times S^2$ (the Cartesian product of two spheres).

(c) Prove that the special linear group

$$SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\}$$

is a differentiable submanifold of $M_n(\mathbb{R}) = \mathbb{R}^{n \times n}$. What is its dimension?

(d*) Prove that the orthogonal group $O(n)$ is a differentiable submanifold of $M_n(\mathbb{R})$. What is its dimension?

Remark. The subsets of $GL_n(\mathbb{R})$ that are both subgroups and submanifolds are called the *classical groups*. These are examples of Lie groups (in fact, the most important ones). Examples include $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $O_n(\mathbb{R})$, $SO_n(\mathbb{R})$, $U_n(\mathbb{C})$, $SU_n(\mathbb{C})$ and $Sp_{2n}(\mathbb{R})$ (the symplectic group).

7.5 In this exercise, we will study submanifolds that arise as level sets of quadratic functions

(a) Recall what a quadratic form Q on a vector space is.

(b) Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form. Prove that Q is differentiable. What is its differential at a point $x \in \mathbb{R}^n$?

(c) What does Sylvester's theorem from linear algebra state? What is the *signature* of a quadratic form? What does it mean for Q to be *non-degenerate*?

(d) Prove that if $Q : \mathbb{R}^n \to \mathbb{R}$ is a non-degenerate quadratic form, then the hypersurface $Q^{-1}(c)$ is a submanifold of $\mathbb{R}^n$ for all $c \neq 0$. What is its dimension?

(e) Is the set

$$S_0(Q) = \{x \in \mathbb{R}^n \mid Q(x) = 0\}$$

a submanifold? The set $S_0(Q)$ is called the *isotropic cone* of the quadratic form Q.

(f) The hypersurfaces

$$S_+(Q) = \{x \in \mathbb{R}^n \mid Q(x) = +1\}, \quad S_-(Q) = \{x \in \mathbb{R}^n \mid Q(x) = -1\}$$

are called the positive and negative *indicatrices* of the quadratic form Q. Show that Q is completely determined by the two indicatrices and the isotropic cone, i.e., if Q_1 and Q_2 are two quadratic forms on $\mathbb{R}^n$ such that

$$S_0(Q_1) = S_0(Q_2), \quad S_+(Q_1) = S_+(Q_2), \quad S_-(Q_1) = S_-(Q_2),$$

then $Q_1 = Q_2$.

B. Bonus exercise:

7.6 Let $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$, where $\{\infty\}$ is an additional point not belonging to $\mathbb{R}^n$. Define a topology on $\widehat{\mathbb{R}}^n$ such that $\mathbb{R}^n$ is open with the usual topology, and the neighborhoods of ∞ are sets of the form $\mathbb{R}^n \setminus K$, where K is compact in $\mathbb{R}^n$.

Now consider the map $f : \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$ defined by

$$f(x) = \begin{cases} \infty, & \text{if } x = p, \\ p, & \text{if } x = \infty, \\ p + k\dfrac{x-p}{\|x-p\|^2}, & \text{if } x \notin \{p, \infty\}, \end{cases}$$

where p is a point in $\mathbb{R}^n$ and $k > 0$ is a real constant. This map is called the *inversion* of center $p \in \mathbb{R}^n$ and modulus $k > 0$. It plays an important role in geometry and analysis; it has the property that the image of any sphere of $\mathbb{R}^n$ not passing through p is a sphere and the image of any sphere passing through p is a hyperplane.

Answer the following questions:

(a) Describe all convergent sequences in $\widehat{\mathbb{R}}^n$ (no need for a formal proof, just an explanation).

(b) Describe the set of fixed points of f, i.e. $\{x \in \widehat{\mathbb{R}}^n \mid f(x) = x\}$.

(c) Prove that f is a homeomorphism of $\widehat{\mathbb{R}}^n$. What is its inverse? Also prove that f restricts to a diffeomorphism of $\mathbb{R}^n \setminus \{p\}$ onto itself.

(d) Prove that if $n = 2$, f defines an anti-holomorphic map on $\mathbb{C} \setminus \{p\}$.

(e) Compute the differential $df_x(h)$ at a point $x \in \mathbb{R}^n \setminus \{p\}$.

(f) Prove that f is a conformal map on $\mathbb{R}^n \setminus \{p\}$ (a map is called conformal if it preserves angles; concretely, this means proving that df_x is a similarity transformation on $\mathbb{R}^n$). What is the similarity ratio of $df_x(h)$?

(g*) Using (without proof, though you can try also establishing this fact) that the inversion maps spheres not passing through p to spheres not passing through p as well as spheres through p to hyperplanes (with two spheres which are tangent at p being mapped to parallel hyperplanes), can you perform the following construction: Exhibit 9 spheres in $\mathbb{R}^3$, with the property that each of them is tangent to at least 5 of the other spheres, and 3 of those spheres are tangent to at least 7 of the other spheres. Note that two surfaces are tangent to each other if, at their points of intersection, they have the same tangent plane.